Maxwell equations

Let's briefly remind ourselfs with physical meaning of Maxwell's equations: electro-magnetic field in vacuum in every spatial point $\bar{r}$ and time t will be determined by two vectors: $E(\bar{r},t)$ and $B(\bar{r},t)$. We can simply write the force that experience point charge q, that moves with speed v:

$$\bar{F} = q\overset{1}{\bar{E}} + q[\bar{v}\overset{2}{\bar{B}}]$$

this is Lorentz force.

What Lorentz force tells us:

① Electric field can be found through the force that experience probe charge when it is static.

② Magnetic field $\bar{B}(\bar{r},t)$ is only determined the part of $\bar{F}$ that appears when the charge is moving.

What can serve as the source of electro-magnetic radiation?

① ρ - density of charge

② $\bar{j}$ - vector of current density.

Maxwell's equations:

$$\operatorname{div}\bar{E} = \frac{\rho}{\varepsilon_0} \qquad \operatorname{div}\bar{B} = 0$$

$$c \operatorname{rot}\bar{B} - \frac{\partial \bar{E}}{\partial t} = \frac{1}{\varepsilon_0}\bar{j} \qquad \operatorname{rot}\bar{E} + \frac{\partial B}{\partial t} = 0$$

Little math walkaround

① div ? del or nabla

$$\bar{f}(x, y, z) = f_x \bar{e}_x + f_y \bar{e}_y + f_z \bar{e}_z$$

$$\operatorname{div} \bar{f} = (\bar{\nabla} \bar{f}) = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

Meaning:

- divergence is the measure of vector field's increase in the direction it points
- measure of the field's tendency to diverge from that point

② rot ?

$$\operatorname{rot} \bar{f} = [\bar{\nabla} \bar{f}] =$$

$$= \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) \bar{e}_x + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) \bar{e}_y +$$

$$+ \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) \bar{e}_z$$

$$\operatorname{rot} \bar{f} = [\bar{\nabla} \bar{f}] = \begin{pmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{pmatrix}$$

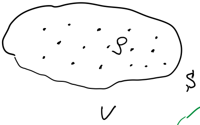
First pair of Maxwell equations

$$\begin{cases} \operatorname{div} \bar{E} = \dfrac{\rho}{\varepsilon_0} & (1.1) \\ c^2 \operatorname{rot} \bar{B} - \dfrac{\partial \bar{E}}{\partial t} = \dfrac{1}{\varepsilon_0} \bar{j} & (1.2) \end{cases}$$

Here c is electro-magnetic constant that is

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$\boxed{1.1}$ Let's take volume V, with closed surface S:

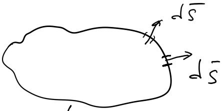


$$\int_V \operatorname{div} \bar{E} = \frac{1}{\varepsilon_0} \int_V \rho \, dV$$

Homework:

Ostrogradsky - Gauss theorem

$$\underbrace{\oint_S \bar{E} \, d\bar{S} = \frac{1}{\varepsilon_0} \int \rho \, dV}_{\text{this is Gauss law}}$$



What Gauss law tells us:

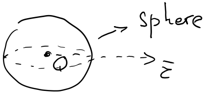
The flux of electric field through closed surface S is determined by the total charge that is confined in the closed volume V.

It will be simple to demonstrate that in case of static, not moving charges it is Coloumb Law.

Strictly mathematically speaking it is not possible as we do not have information on rot $\bar{E}$. However, if we assume stationary point charge:

$$\oint_S \bar{E}\, d\bar{S} = \frac{1}{\varepsilon_0}\int_V \rho\, dV = \frac{Q}{\varepsilon_0}$$

$$4\pi r^2 (\bar{r}\,\bar{E}) = \frac{Q}{\varepsilon_0}$$



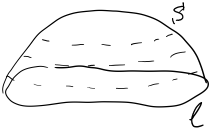
$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\bar{r} \longrightarrow \text{Coloumb law}$$

what about dimensionality?

(1.2)

$$c^2\, \mathrm{rot}\, \bar{B} - \frac{\partial \bar{E}}{\partial t} = \frac{1}{\varepsilon_0}\bar{j} \qquad (1,2)$$

Now we take closed contour l and open surface S that is supported by l:



Let's calculate the flux of (1.2) left and right parts through surface S

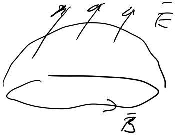
$$\int_S \left(c^2 \operatorname{rot} \bar{B} - \frac{\partial E}{\partial t}\right) d\bar{S} = \frac{1}{\varepsilon_0} \int_S \bar{j}\, d\bar{S}$$

Homework:
Stock's
theorem

$$c^2 \oint_\ell \bar{B}\, d\bar{\ell} = \underbrace{\frac{1}{\varepsilon_0} \int_S \bar{j}\, d\bar{S}}_{1} + \underbrace{\frac{\partial}{\partial t} \int_S \bar{E}\, d\bar{S}}_{2}$$

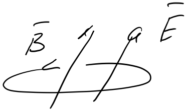
What each term means:

① Circulation of vector $\bar{B}$ on any closed contour is determined by total current I through the surface S



Ampere's law / Biot-Savart law

It means the changing electric field $\bar{E}$ $\left(\frac{\partial \bar{E}}{\partial t}\right)$ creates vortex magnetic field.



② Maxwell brought up the new term - displacement current $\frac{\partial E}{\partial t}$. Why?

This appeared to be a crucial point to bring unified theory of
$$\begin{cases}\text{Electricity}\\ \text{Magnetism}\\ \text{Optics}\end{cases}$$

There were historically two types of current:

1) conduction current $\bar{j}$.
This is directional movement of electric charges on long distances.

$$\bar{e}\ \circ \longrightarrow$$

We can easily measure it in closed circuit by ampere-meter.

2) So-called current from bound charges - the average displacement of charges (electrons, nuclears) compared to molecule center.

The main argument was to explain possibility of registration of current for open circuit if alternative current is used.

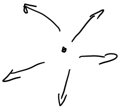
Bringing $\frac{\partial E}{\partial t}$ Maxwell made sure that the charge conservation law has been fulfiled.

Let's show it:

We know (1.2) $c^2 \operatorname{rot}\bar{B} - \frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0}\bar{j}$

Let's see how it diverges from critical point, i.e.:

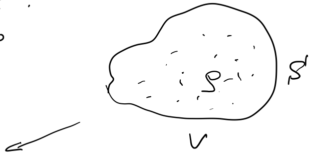
$$\operatorname{div}\left(c^2 \underbrace{\cancel{\operatorname{rot}\bar{B}}}_{0}\right) - \frac{\partial}{\partial t}\operatorname{div}\bar{E} = \frac{1}{\varepsilon_0}\operatorname{div}\bar{j}$$



$$-\frac{\partial}{\partial t}\operatorname{div}\bar{E} = \frac{1}{\varepsilon_0}\operatorname{div}\bar{j}$$

Since $\operatorname{div}\bar{E} = \frac{\rho}{\varepsilon_0}$:

$$-\frac{\partial \rho}{\partial t} = \operatorname{div}\bar{j}$$



$$\frac{\partial}{\partial t}\int_V \rho\, dV = -\oint_S \bar{j}\, d\bar{S} \longrightarrow$$ Charge conservation law

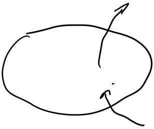
The change of the total charge in closed volume V can be only done by carrying it through the surface S, that defines this volume.

Second pair of Maxwell's equations

$$\operatorname{div}\bar{B} = 0 \quad (2.1)$$

$$\operatorname{rot}\bar{E} + \frac{\partial \bar{B}}{\partial t} = 0 \quad (2.2)$$

(2.1) $\operatorname{div}\bar{B} = 0$



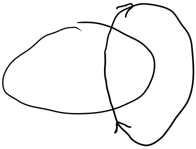
$$\oint \bar{B}\, d\bar{S} = 0$$

S The flux of magnetic field through any closed surface S is 0.

What does it mean?

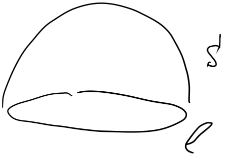
It means that $\bar{B}$ is closed contour

Gauss law of magnetism



(2.2) $$\mathrm{rot}\,\bar{E} + \frac{\partial \bar{B}}{\partial t} = 0$$

$$\int_S \mathrm{rot}\,\bar{E}\,d\bar{S} + \int_S \frac{\partial \bar{B}}{\partial t}\,d\bar{S} = 0$$



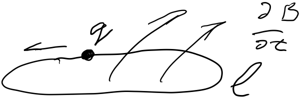
$$\oint_\ell \bar{E}\,d\bar{\ell} = -\frac{\partial}{\partial t}\int_S \bar{B}\,d\bar{S} \longrightarrow \text{Faraday Law}$$

Faraday experimentally demonstrated that electromotive force (EMF) is defined by the speed of magnetic flux change through surface S:

$$\boxed{\oint_\ell \bar{E}\,d\bar{\ell}} = \underbrace{\frac{\partial}{\partial t}}_{\text{speed}}\underbrace{\boxed{\int_S \bar{B}\,d\bar{S}}}_{\text{flux}}$$

EMF

EMF is defined as electromagnetic work done on a unit charge when it has travelled one full round in closed loop.



In other words, varying magnetic fields lead to varying electric fields and vice versa.

Finally, Maxwell's equations:

$$\mathrm{div}\,\bar{E} = \frac{\rho}{\varepsilon_0}$$

$$\mathrm{div}\,\bar{B} = 0$$

$$c^2\,\mathrm{rot}\,\bar{B} - \frac{\partial \bar{E}}{\partial t} = \frac{1}{\varepsilon_0}\bar{j}$$

$$\mathrm{rot}\,\bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

We will add so-called material equations:

$$\bar{D} = \varepsilon\varepsilon_0\bar{E} \qquad \bar{B} = \mu\mu_0\bar{H}$$